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A q-deformed completely integrable Bose gas model

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Abstract. We construct the Hamiltonian of a new quantum integrable 'q-boson' lattice model in $1+1$ dimensions which has q-bosons as dynamical variables and solve it for its energy eigenstates and energy eigenvalues under periodic boundary conditions of finite period. This model can be regarded as the q-deformation of the integrable lattice Bose gas (the quantum lattice nonlinear Schrödinger (quantum lattice NLS)) model. In appropriate continuum limits both of these lattice models become the quantum repulsive NLS model (the Bose gas). In the lattice case the distinction between the q-bosons and ordinary bosons leads to significant new features—particularly in the energy spectrum for which the q-boson lattice model has an optical branch and bound state spectra as well as the (one) acoustic branch of the quantum lattice NLS model. It is argued that this sort of distinction is generic for discrete systems with a finite number of degrees of freedom as opposed to continuum field theories like the Bose gas where quantization by quantum group quantization or by canonical quantization leads to the same results.

Various properties of the q-bosons are given and used. In particular a q-boson Primakov–Holstein transformation is given which connects the q-boson lattice model with the XXZ models of arbitrary spin.

1. Introduction

The quantum inverse scattering method (QISM) ([1–3] and references therein) has provided fundamental insights into the theory of quantum integrable systems in one space and one time $(1+1)$ dimensions. The method showed how the Bethe equations for a quantum integrable model could be derived, and the system solved for its eigenvectors and eigenvalues, by what are essentially algebraic methods without the explicit ansatz of the Bethe method. The formulation of the commutation relations for the elements of the quantum monodromy matrix $\hat{T}(\lambda)(\lambda \in \mathbb{C})$ in terms of the quantum R-matrix, and the existence of the Yang–Baxter equations for the R-matrix [1–3], has led to the realization that the algebra generated by the elements of $\hat{T}(\lambda)$ has the additional structure of a co-algebra [4, 5]. Consequently at special values of the spectral parameter λ (notably $\lambda=\pm\infty$ [4]) the elements of $\hat{T}(\lambda)$ form a Hopf algebra, a non-commutative and non-co-commutative Hopf algebra [6]. In the context of quantum integrable models these Hopf algebras are called 'quantum groups' [4–7]. In terms of duals [5] these quantum groups can also be regarded as the *deformations* (q-deformations) of the universal enveloping algebra of an underlying classical Lie group [5–7]. Deformation of the loop algebras [5] and recently of the affine Lie algebras [8] allows

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an extension of the deformed algebras to Hopf algebras for arbitrary λ, and all of these algebras are now loosely called 'quantum groups'.

The origins of the deformed algebras can be traced to the theory of Lie admissible algebras [9] and Umbral calculus [10], and these algebras have already appeared in hadron physics [11]. The deformed analogue of the harmonic oscillator has already been studied in [12]. The representation theory of the quantum groups and deformed algebras is currently under intensive investigation, and in particular several realizations of the q-deformed Heisenberg algebra (a 'q-oscillator' algebra [13]) have been rederived [13–16]. This q-deformation of the Heisenberg algebra is natural because it is connected with the q-deformation of the Lie algebra $\mathrm{su}(2) \to \mathrm{su}_q(2)$ [4–7] as we shall show in section 2 of this paper. This paper is concerned with the q-deformed bosons satisfying this deformed Heisenberg algebra. We shall call them 'q-bosons' in here.

Quantum groups and algebras arise in many problems of current interest in mathematical physics (knot theory [17], quantum symmetries [18], and see section 3) and the quantum integrable models are notable examples. However, the important questions surely concern their physics. Is the algebraic basis of the quantum groups equivalent to the usual canonical quantization, and if not are there physical situations where these two alternative modes of quantization make different predictions which can be tested by experiment? For the quantum nonlinear (and linear) field theories in $1+1$ dimensions quantization by the quantum groups is equivalent to their canonical quantization [5]. *On the other hand, simple models based on the q-bosons [16, 19–21] lead to predictions* different from that of conventional quantum mechanics.

We point out in this paper that the equivalence of quantum group and canonical quantization for the field theories in $1+1$ actually rests on the continuous nature of these field theories. Thus a quantum completely integrable lattice quantized by the quantum groups can show up new features. We construct a quantum completely integrable lattice in $1+1$ in this paper which has q-boson fields as its dynamical variables and show indeed that it exhibits new features.

The model is evidently a q-deformation of the quantum nonlinear Schrödinger (quantum NLS) model on the lattice [22, 23] and the q-deformation introduces new features. The quantum NLS model on the lattice [22, 23] is the 'lattice Bose gas'. We therefore call the model constructed in this paper 'the q-Bose gas model'. As remarked, we show that this q-Bose gas model is intrinsically different from the lattice Bose gas. On the other hand, we show that a natural continuum limit of the model is identical with the quantum NLS model (the 'ordinary' Bose gas [24]) for which [3, 5] quantum group quantization and canonical quantization are equivalent.

The paper is organized as follows. After a brief review in section 2 of the algebra of q-bosons we turn in the section 3 to the q-deformation of the lattice NLS model [23]. For lattices the fundamental object is the 'local' operator $\hat{L}_n(\lambda)$ (local transition matrix) *from which the quantum* monodromy matrices $\hat{T}(\lambda)$ can be constructed. The trace of the monodromy matrix $\hat{T}(\lambda)$ is the generating function of the integrals of motion (this is true in both the quantum and classical cases [1, 3]). Thus in section 3 we give the $\hat{L}_n(\lambda)$ operator of the q-deformed model (i.e. of the 'q-Bose gas model') and in section 4 we derive the Hamiltonian of the model. The spectrum is found in section 5 by means of the algebraic Bethe ansatz (i.e. by the QISM). In section 6 we take particular continuum limits of the model and regain the 'ordinary' Bose gas of [24]. A number of connected problems, notably calculation of the correlation functions and critical exponents [25] and investigation of certain bound states, are deferred to following papers.

2. The q-bosons of the q-Bose gas model

The q-Bose algebra is generated [13–16] by three elements $a^\dagger$, a and N $(=N^\dagger)$. These are a q-creation operator $a^\dagger$, a q-annihilation operator $a=(a^\dagger)^\dagger$ and the number operator N defined to satisfy

$$[N, a^\dagger]=a^\dagger \qquad [N, a]=-a \tag{2.1}$$

and

$$aa^\dagger - qa^\dagger a = q^{-N} \tag{2.2}$$

where q is a c-number taken here to be $q=e^{i\gamma}$ or $q=e^\gamma$, with $\gamma\in\mathbb{R}$, a real number (we shall not need the general case $\gamma\in\mathbb{C}$).

For $q=e^\gamma$ one can construct the representation of the relations (2.1) and (2.2) in the Fock space spanned by the normalized eigenstates of the number operator N in the usual way. The q-boson vacuum $|0\rangle$ is defined by

$$a|0\rangle=0 \tag{2.3}$$

and the normalized state vectors $|n\rangle$ are constructed as usual so that

$$N|n\rangle=n|n\rangle \qquad n=1,2,\ldots. \tag{2.4}$$

We can put $|n\rangle$ in the form [15]

$$|n\rangle=([n]!)^{-1/2}(a^\dagger)^n|0\rangle \tag{2.5a}$$

in which

$$[n]!=[1],[2],\ldots,[n] \tag{2.5b}$$

where

$$[n]\equiv(q^n-q^{-n})/(q-q^{-1}) \tag{2.5c}$$

and, for $q=e^\gamma>0, [n]>0$. Evidently $[n]!$ defined by equations (2.5*b, c*) is the usual $n!$ for $q\to1$ $(\gamma\to0)$. We shall refer to the notation $[n]$ as the 'box' notation. The 'box' notation in $[n]$ is defined by (2.5*c*). Since, on the Fock space spanned by the $|n\rangle$, (2.2) is invariant under $q\to q^{-1}$,

$$aa^\dagger - q^{-1}a^\dagger a = q^N \tag{2.6}$$

on that Fock space; and it follows from (2.2) and (2.6) that

$$a^\dagger a=[N]\equiv(q^N-q^{-N})/(q-q^{-1}) \tag{2.7a}$$

and

$$aa^\dagger=[N+1] \tag{2.7b}$$

on that Fock space. In relations (2.7) the 'box' notation is now extended to the case of operators, N and $N+1$. These several results mean that the Fock space spanned by the $|n\rangle$, equation (2.5*a*), forms a natural extension of the normal Fock space $|n\rangle$ with $[n]!$ replacing $n!$ and $[n]$ replacing n appropriately. As $q\to1$ $(\gamma\to0)$, relations (2.1), (2.2) and (2.6) and (2.7) form the usual Bose algebra, $N=a^\dagger a$, and the states $|n\rangle$, (2.5*a*) are the usual $|n\rangle$. however, we can show, by using the representation of the operators in terms of ordinary boson operators, equations (6.14) below, that we can also construct the Fock space defined by (2.5*a*) in terms of ordinary boson states $|n\rangle$.

For the q-Bose gas model constructed in this paper we use $q=e^{i\gamma}$, $\gamma \in \mathbb{R}$. In this case relations (2.6) and (2.7a, b) follow immediately as operator relations. However, for $n > \pi\gamma^{-1}$, $[n]$ may be <0. We use $q = e^{i\gamma}$, $0<\gamma<\frac{1}{2}\pi$ for the q-bose gas model in section 5. Then by making an appropriate choice for the representation of the algebra defined through (2.1) and (2.2) we can still construct the Fock space. Thus there is no problem with $q=e^{i\gamma}$ in this regard. Note that the notation which is (2.7a, b) for operators is used throughout the paper.

The quantum enveloping algebra $\mathrm{su}_q(2)$ of the Lie algebra su(2) is the algebra with generators S^z, $S^\pm$ and relations

$$[S^\pm, S^z] = \mp S^\pm \qquad [S^+, S^-] = [2S^z] \equiv \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}. \tag{2.8}$$

The su(2) Lie algebra has the well known Primakov–Holstein representation in terms of Bose operators. The corresponding Primakov–Holstein representation for $\mathrm{su}_q(2)$ in terms of q-boson operators (2.1), (2.2) is

$$S^+ = \sqrt{[\alpha - N]}\,a \qquad S^- = a^\dagger\sqrt{[\alpha - N]} \qquad S^z = \tfrac{1}{2}\alpha - N \tag{2.9}$$

in which α is an arbitrary complex c-number (the notation (2.5b) is again used here and it will be used throughout the paper).

For the q-Bose gas we need $M>1$ independent, mutually commuting, q-bosons $a_n^\dagger$, a_n, N_n with $n = 1, 2, \ldots, M$. Relations (2.1) and (2.2) now become

$$[N_n, a_m^\dagger] = a_n^\dagger \delta_{mn} \qquad [N_n, a_m] = -a_n\delta_{mn}$$

and

$$a_n a_m^\dagger - \delta_{mn} q a_m^\dagger a_n = \delta_{mn} q^{-N_n} + (1-\delta_{mn}) a_m^\dagger a_n. \tag{2.10}$$

The normalized state vectors are

$$|0\rangle = \prod_{j=1}^{M} |0\rangle_j \qquad |n\rangle = \prod_{j=1}^{M} |n_j\rangle = \prod_{j=1}^{M} ([n_j]!)^{-1/2} (a_j^\dagger)^{n_j} |0\rangle \tag{2.11}$$

and define the extended Fock space. In a continuum limit in which $n\delta \to x$, $a_n \to \sqrt{\delta}\,\beta(x)$, $a_n^\dagger \to \sqrt{\delta}\,\beta^\dagger(x)$ and $N_n \to \delta N(x)$, as the 'lattice spacing' $\delta \to 0$, one finds from the operator relation (2.10) that

$$[\beta(x), \beta^\dagger(y)] = \delta(x-y) \tag{2.12}$$

together with $N(x) = \beta^\dagger(x)\beta(x)$ and equations (2.12) are the usual canonical commutation relations. Thus insofar as this result for the q-bosons is generic we expect q-quantization to coincide with canonical quantization in a continuum limit—that is, we expect the two quantization to coincide for all field theories, linear or nonlinear.

3. The q-deformation of the quantum NLS model on a lattice

We consider a periodic chain with M sites. The q-Bose operators $a_j^\dagger$, a_j, N_j forming the M independent dynamical variables (2.10) are assigned successively, $j = 1, 2, \ldots, M$, to the sites j; $a_{j+M}^\dagger = a_j^\dagger$, $a_{j+M} = a_j$, $N_{j+M} = N_j$.

A most general model of interacting q-bosons on such a lattice can now be introduced through the following $\hat{L}_n$-operator:

$$\hat{L}_n(\lambda)=\begin{bmatrix}[N_n-\frac{1}{2}\alpha_n-\mathrm{i}\lambda] & -\mathrm{i}a_n^\dagger\sqrt{[N_n-\alpha_n]}\\ \mathrm{i}\sqrt{[N_n-\alpha_n]}a_n & [N_n-\frac{1}{2}\alpha_n+\mathrm{i}\lambda]\end{bmatrix} \tag{3.1}$$

where the notation (2.5*b*) is again used. In $\hat{L}_n(\lambda)\lambda\in\mathbb{C}$ is a spectral parameter and α_n is a complex number which in general will depend on the lattice site (now labelled n). Reference to [22] shows that this $\hat{L}_n$-operator can be considered to be the q-deformation of the $\hat{L}_n$-operator for the quantum nonlinear Schrödinger model on a lattice (QLNS model). When $q\to 1$, (3.1) becomes exactly the $\hat{L}_n$-operator of the QLNS model.

Alternatively, (3.1) derives from the $\hat{L}_n$-operator of the XXZ Heisenberg chain of arbitrary spin [26, 27]:

$$\begin{bmatrix}-[\mathrm{i}\lambda+S_n^z] & -S_n^-\\ S_n^+ & [\mathrm{i}\lambda-S_n^z]\end{bmatrix}=-\sigma^3\hat{L}_n^{XXZ}(\lambda) \tag{3.2}$$

(σ^3 is a Pauli matrix) by making the Primakov–Holstein transformation (2.9). Although $S_n^\pm$, S_n^z are elements of the quantum group $\mathrm{su}_q(2)$ they arise naturally in the definition of the $\hat{L}_n$-operator of the XXZ magnetic chain of arbitrary spin and these chains have the usual canonical quantization. We shall report on this situation elsewhere.

As is generally the case in the QISM [1-3] the $\hat{L}_n$-operators (3.1) or (3.2) are intertwined in terms of an R-matrix: one finds, with $\otimes$ meaning Kronecker product, that

$$R(\lambda-\mu)\hat{L}_n(\lambda)\otimes\hat{L}_n(\mu)=\hat{L}_n(\mu)\otimes\hat{L}_n(\lambda)R(\lambda-\mu) \tag{3.3}$$

where $R(\lambda,\mu)$ $(=R(\lambda-\mu))$ is the same for either (3.1) or (3.2) and is given by

$$R(\lambda,\mu)=\begin{bmatrix}f(\mu,\lambda) & 0 & 0 & 0\\ 0 & g(\mu,\lambda) & 1 & 0\\ 0 & 1 & g(\mu,\lambda) & 0\\ 0 & 0 & 0 & f(\mu,\lambda)\end{bmatrix}. \tag{3.4}$$

This is a trigonometric R-matrix (the XXZ R-matrix) with

$$f(\mu,\lambda)=\frac{[\mathrm{i}(\lambda-\mu)+1]}{[\mathrm{i}(\lambda-\mu)]}\qquad g(\mu,\lambda)=\frac{1}{[\mathrm{i}(\lambda-\mu)]}. \tag{3.5}$$

This R-matrix depends on q and becomes the rational R-matrix of the QLNS and continuum quantum NLS models only when $q\to 1$.

We shall make use of the following properties of the $\hat{L}_n$-operator (3.1):

$$\hat{L}_n(-\lambda)=\sigma^1\hat{L}_n^t(\lambda)\sigma^1 \tag{3.6}$$

$$\mathrm{e}^{\beta N_n}\hat{L}_n(\lambda)\,\mathrm{e}^{\frac{1}{2}\beta\sigma^3}=\mathrm{e}^{\frac{1}{2}\beta\sigma^3}\hat{L}_n(\lambda)\,\mathrm{e}^{\beta N_n} \tag{3.7}$$

in which the σ^i are Pauli matrices, N_n is defined by (2.10) and β is a c-number. We also need the quantum determinant of $\hat{L}_n(\lambda)$ which is defined [2] through

$$\begin{aligned}\hat{I}\,\mathrm{Det}_q\,\hat{L}_n(\lambda)&=\hat{L}_n(\lambda)\sigma^2\hat{L}_n^t(\lambda+\mathrm{i})\sigma^2\\ &=-\hat{I}([\mathrm{i}\lambda+\tfrac{1}{2}\alpha_n][\mathrm{i}\lambda-1-\tfrac{1}{2}\alpha_n])\end{aligned} \tag{3.8}$$

where $\hat{I}$ is the unit matrix.

We can now trace out the usual steps of the QISM [1-3]. The quantum monodromy matrix $\hat{T}(\lambda)$ is introduced as

$$\hat{T}(\lambda)=\widehat{\prod_n}\hat{L}_n(\lambda)\equiv\hat{L}_M(\lambda)\ldots\hat{L}_2(\lambda)\hat{L}_1(\lambda)$$
$$=\begin{bmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{bmatrix}. \tag{3.9}$$

The commutation relations of the elements A, B, C, D of $\hat{T}(\lambda)$ are then given by

$$R(\lambda-\mu)\hat{T}(\lambda)\otimes\hat{T}(\mu)=\hat{T}(\mu)\otimes\hat{T}(\lambda)R(\lambda-\mu) \tag{3.10}$$

a result which follows from (3.3) and (2.10) and the commutativity of the elements of the $\hat{L}_n$-operator at the different lattice sites (see (2.10)). The significant quantity (under the chosen periodic boundary conditions) is the matrix trace of the monodromy matrix

$$\hat{\Delta}(\lambda)\equiv\mathrm{Tr}\ \hat{T}(\lambda)=A(\lambda)+D(\lambda). \tag{3.11}$$

The matrix trace of (3.10) means that

$$[\hat{\Delta}(\lambda),\hat{\Delta}(\mu)]=0 \tag{3.12}$$

and ($\lambda,\mu\in\mathbb{C}$) there is a large number of mutually commuting operators $\hat{\Delta}(\lambda)$; the model is therefore quantum integrable. The Hamiltonian of the model, $\hat{H}$, can be represented as a linear combination of the derivatives of $\partial/\partial\lambda(\ln\Delta(\lambda))$ at some set of fixed points $\lambda=\nu_\alpha$. It follows from (3.7) that $\hat{\Delta}(\lambda)$ commutes with the total number operator $\hat{N}$:

$$[\hat{\Delta}(\lambda),\hat{N}]=0 \qquad \hat{N}=\sum_{j=1}^{M}N_j. \tag{3.13}$$

The number operator $\hat{N}$ therefore commutes with the Hamiltonian of the model $[\hat{H},\hat{N}]=0$.

4. The lattice q-Bose gas

We now construct the Hamiltonian of the integrable q-boson lattice model ('the q-Bose gas') which can be considered to be the q-deformation of the QLNS [22, 23] as explained. We recall that $q=\mathrm{e}^{\mathrm{i}\gamma}$ or $q=\mathrm{e}^{\gamma}$ with $\gamma\in\mathbb{R}$.

If $\hat{H}$ is to be Hermitian we need to evaluate $\partial/\partial\lambda(\ln\hat{\Delta}(\lambda))$ at fixed points ν_α and at ν_α^*, their complex conjugates. A calculation of $\hat{T}(\lambda)$, (3.9), using the $\hat{L}_n(\lambda)$ of (3.1) in which the c-numbers α_n do not specifically depend on the site n, leads to a non-factorizable $\hat{\Delta}(\lambda)\equiv\mathrm{Tr}\ \hat{T}(\lambda)$, a non-local $\partial(\ln\hat{\Delta}(\lambda))/\partial\lambda$ at $\lambda=\nu_n^*$ and a non-local $\hat{H}$—simply because operators at the same site do not commute. The same problem arose in the undeformed QLNS model and it is surmounted by distinguishing the odd and even lattice sites. We therefore set α_n in the $\hat{L}_n$-operator (3.1) to $-\alpha_n=\Lambda+(-1)^n$. The homogeneous, i.e. independent of the lattice site n, parameter $\Lambda\in\mathbb{R}$ ($\Lambda>0$), and it will prove (section 6) to have the sense of an inverse lattice spacing δ^{-1}. We therefore choose

$$\hat{L}_n(\lambda)=\begin{bmatrix}[N_n+\frac{1}{2}\Lambda+\frac{1}{2}(-1)^n-\mathrm{i}\lambda] & -\mathrm{i}a_n^\dagger\rho_n\\ \mathrm{i}\rho_n a_n & [N_n+\frac{1}{2}\Lambda+\frac{1}{2}(-1)^n+\mathrm{i}\lambda]\end{bmatrix} \tag{4.1}$$

in which

$$\rho_n = \sqrt{[N_n + (-1)^n + \Lambda]}. \tag{4.2}$$

The quantum determinant of this $\hat{L}_n$-operator is

$$\hat{I}\ \mathrm{Det}_q\, \hat{L}_n(\lambda) = -([\mathrm{i}\lambda - \mathrm{i}\nu_1^{(n)}][\mathrm{i}\lambda - \mathrm{i}\nu_2^{(n)}])\hat{I} \tag{4.3}$$

where

$$\nu_1^{(n)} = -\mathrm{i}\tfrac{1}{2}(\Lambda + (-1)^n) \qquad \nu_2^{(n)} = -\mathrm{i}(1 - \tfrac{1}{2}(\Lambda + (-1)^n)). \tag{4.4}$$

The Hamiltonian $\hat{H}$ is found as a linear combination of the derivatives of $\ln \hat{\Delta}(\lambda) \equiv \ln \mathrm{Tr}\ \hat{T}(\lambda)$, equation (3.11). We express $\hat{H}$ in terms of lattice densities $\hat{H}_n$:

$$\hat{H} = \sum_{n=1}^{M} \hat{H}_n \tag{4.5}$$

and we shall call $\hat{H}$ local if its densities $\hat{H}_n$ depend on the dynamical variables $a_j^\dagger$, a_j, N_j in a certain neighbourhood of the nth site $(n-m) \leqslant j \leqslant n+l$, $(m+l) < \infty$. This Hamiltonian then describes in each $\hat{H}_n$ a direct interaction between $(m+l+1)$ neighbours.

The locality of $\hat{H}$ is guaranteed since [22] the $\hat{L}_n$-operator (and therefore $\hat{T}(\lambda)$) is proportional to the 1D projectors at the points $\nu = \nu_1^{(n)}$, $\nu_1^{*(n)}$, $(\nu_2^{(n)}, \nu_2^{*(n)})$ of (4.4) where $\mathrm{Det}_q\, \hat{L}_n(\lambda)$ vanishes. To be more precise we introduce a two-component column vector g whose components g_1, g_2 are quantum operators. At the odd sites, where $n = 1$ (mod 2),

$$g_1(n) = -\mathrm{i}a_n^\dagger \qquad g_2(n) = \sqrt{[N_n + \Lambda - 1]} \tag{4.6}$$

at the even sites, where $n = 0$ (mod 1),

$$g_1(n) = -\mathrm{i}a_n^\dagger \qquad g_2(n) = \sqrt{[N_n + \Lambda]}. \tag{4.7}$$

We pick out the point

$$\lambda = \nu = -\mathrm{i}\tfrac{1}{2}(\Lambda - 1) \tag{4.8}$$

where $\mathrm{Det}_q\, \hat{L}_n(\lambda)$, equation (4.3), vanishes, to see that at these odd sites the $\hat{L}_n$-operator is the 'direct' projector with elements

$$(\hat{L}_n(\nu))_{ik} = g_i(n) g_k^\dagger(n) \qquad i, k = 1, 2 \tag{4.9}$$

while at the even sites it is this projector in reverse order

$$(\hat{L}_n(\nu))_{ik} = g_k^\dagger(n) g_i(n) \qquad i, k = 1, 2. \tag{4.10}$$

(Evidently if we introduce the column vector $g = (g_1, g_2)^T$ and the scalar product $(g^\dagger g) = g_1^\dagger g_1 + g_2^\dagger g_2$, $g_i g_k^\dagger$ is a projector up to normalization in the sense that $\hat{L}_n g = g(g^\dagger g)$, $\hat{L}_n^2 g = g(g^\dagger g)^2$, etc.)

The point $\lambda = \nu^*$ is a conjugated zero of $\mathrm{Det}_q\, \hat{L}_n(\lambda)$:

$$\nu^* = -\nu = \mathrm{i}\tfrac{1}{2}(\Lambda - 1). \tag{4.11}$$

Thus the $\hat{L}_n$-operator is the reverse-order projector on the even sites with elements

$$(\hat{L}_n(\nu^*))_{ik} = (g^\dagger(n)\sigma^1)_i (\sigma^1 g(n))_k \qquad i, k = 1, 2 \tag{4.12}$$

(σ^1 is the Pauli matrix) and at the odd sites it is the direct projector

$$(\hat{L}_n(\nu^*))_{ik} = (\sigma^1 g(n))_k (g^\dagger(n)\sigma^1)_i. \tag{4.13}$$

We assume that the number M of points in one period of the lattice is even. By using (4.9) and (4.10) in (3.9) we find that $\hat{\Delta}(\nu)$ is given by

$$\hat{\Delta}(\nu)=(g^{\dagger}(M)g(M-1))\prod_{n=1}^{\frac{1}{2}M-1}\{(g^{\dagger}(2n)g(2n-1))(g^{\dagger}(2n+1)g(2n))\}(g^{\dagger}(1)g(M)) \tag{4.14}$$

which is factored (the parentheses indicate the scalar product $(g^{\dagger}g)=(g_1^{\dagger}g_1+g_2^{\dagger}g_2)$). The generating function for $\hat{H}$, $\ln\hat{\Delta}(\nu)$, is therefore

$$\ln\hat{\Delta}(\nu)=\ln\prod_{n=1}^{M}\Delta_n. \tag{4.15}$$

The first derivative of $\ln\hat{\Delta}(\lambda)$ at $\lambda=\nu$

$$\frac{\partial}{\partial\lambda}(\ln\hat{\Delta}(\lambda))|_{\lambda=\nu}=\frac{-\mathrm{i}2\ln q}{q-q^{-1}}\sum_{n=1}^{M}\hat{t}_n. \tag{4.16}$$

It is local and the $\hat{t}_n$ depend on the parity of the site. For the even sites

$$\begin{aligned}\hat{t}_n=(g^{\dagger}(n-1)g(n-2))^{-1}\{&(g^{\dagger}(n+1)g(n))^{-1}(g^{\dagger}(n)g(n-1))^{-1}\\ &\times(g^{\dagger}(n+1)G(n)g(n-1))\}(g^{\dagger}(n-1)g(n-2))\end{aligned} \tag{4.17}$$

and for the odd sites

$$\begin{aligned}\hat{t}_n=(g^{\dagger}(n+2)g(n+1))^{-1}\{&(g^{\dagger}(n)g(n-1))^{-1}(g^{\dagger}(n+1)g(n))^{-1}\\ &\times(g^{\dagger}(n+1)G(n)g(n-1)\}(g^{\dagger}(n+2)g(n+1)).\end{aligned} \tag{4.18}$$

The matrix operator $G(n)$ is given by

$$G(n)=\sigma^3\{q^{N_n+1+\frac{1}{2}\sigma^3(1-\Lambda)}+q^{-N_n-1-\frac{1}{2}\sigma^3(1-\Lambda)}\}\qquad n=0\ (\mathrm{mod}\ 2) \tag{4.19}$$

and

$$G(n)=\sigma^3\{q^{N_n+\frac{1}{2}(1-\sigma^3)(\Lambda-1)}+q^{-N_n-\frac{1}{2}(1-\sigma^3)(\Lambda-1)}\}\qquad n=1\ (\mathrm{mod}\ 2). \tag{4.20}$$

A similar expression for $\partial(\ln\hat{\Delta}(\lambda))/\partial\lambda$ will arise at $\lambda=\nu^*$ which is also local. One can prove that

$$\begin{aligned}\partial(\ln\hat{\Delta}(\lambda)/\partial\lambda)|_{\lambda=\nu^*}&=[\partial(\ln\hat{\Delta}(\lambda)/\partial\lambda)|_{\lambda=\nu}]^{\dagger}\\ &=-P\partial(\ln\hat{\Delta}(\lambda)/\partial\lambda)_{\lambda=\nu}P.\end{aligned} \tag{4.21}$$

In the third expression P is the operator of space reflection.

We can now define the Hamiltonian of the model as follows: $\hat{H}\equiv\hat{H}_0$ where

$$\begin{aligned}\hat{H}_0=\frac{q-q^{-1}}{2\ln q}\left\{-\mathrm{i}\frac{\partial}{\partial\lambda}\ln(\hat{\Delta}(\lambda)\Delta_0^{-1}(\lambda))|_{\lambda=\nu}+\mathrm{i}\frac{\partial}{\partial\lambda}(\ln(\hat{\Delta}(\lambda)\Delta_0^{-1}(\lambda))|_{\lambda=\nu^*}\right\}\\ +2\hat{N}/[\tfrac{1}{2}\Lambda+1][\tfrac{1}{2}\Lambda-1].\end{aligned} \tag{4.22}$$

In this $\hat{N}$ is the total number operator (3.13), $\hat{N}=\Sigma_{j=1}^{M}N_j$, and $\Delta_0(\lambda)$ is $\hat{\Delta}(\lambda)$ in the absence of any fields, namely

$$\Delta_0(\lambda)=([\tfrac{1}{2}(\Lambda+1)-\mathrm{i}\lambda][\tfrac{1}{2}(\Lambda-1)-\mathrm{i}\lambda])^{M/2}+([\tfrac{1}{2}(\Lambda+1)+\mathrm{i}\lambda][\tfrac{1}{2}(\Lambda-1)+\mathrm{i}\lambda])^{M/2}. \tag{4.23}$$

In this way we have constructed the Hamiltonian in terms of the original q-Bose fields. It is local in the sense of (4.5) and is given finally as

$$\hat{H}_0=-\sum(\hat{t}_n+\hat{t}_n^{\dagger}+2[2\Lambda-1]/[\Lambda][\Lambda-1])+2\hat{N}/[\tfrac{1}{2}(\Lambda+1)][\tfrac{1}{2}(\Lambda-1)] \tag{4.24}$$

in which $\hat{t}_n$ is defined by (4.17) and (4.18). It is Hermitian and, as the expressions for $\hat{t}_n$ show, involves the direct interaction of four neighbours on the lattice. Since $[\hat{H}_0, \hat{N}]=0$ we can consider $\hat{H}=\hat{H}_0-\bar{\mu}\hat{N}$ where $\bar{\mu}$ is the chemical potential; $\bar{\mu}>0$.

5. Solution of the model

We can solve the model for its eigenenergies and eigenstates by means of the algebraic Bethe ansatz (QISM [1-3]). The state which is annihilated by the lower left element of the $\hat{L}_n$-operator is called the pseudo-vacuum: it is clear that this state $|0\rangle_n$ of the model is the q-Bose vacuum (2.3): $a_n|0\rangle=0$. The generating pseudo-vacuum, namely the vacuum eigenstate of the monodromy matrix $\hat{T}(\lambda)$ (3.9), is then the state $|0\rangle$ of (2.11):

$$|0\rangle=\prod_{n=1}^{M}|0\rangle_n \qquad \hat{T}(\lambda)|0\rangle=\hat{\prod_n}\hat{L}_n(\lambda)|0\rangle_n \tag{5.1}$$

with $\hat{L}_n$ given by (4.1). The vacuum eigenvalues of the elements of $\hat{T}(\lambda)$ appear as

$$A(\lambda)|0\rangle=(a(\lambda))^{M/2}|0\rangle \qquad D(\lambda)|0\rangle=(d(\lambda))^{M/2}|0\rangle \qquad C(\lambda)|0\rangle=0$$

in which

$$\begin{aligned} a(\lambda)&=[\tfrac{1}{2}(\Lambda+1)-\mathrm{i}\lambda][\tfrac{1}{2}(\Lambda-1)-\mathrm{i}\lambda]\\ d(\lambda)&=[\tfrac{1}{2}(\Lambda+1)+\mathrm{i}\lambda][\tfrac{1}{2}(\Lambda-1)+\mathrm{i}\lambda]. \end{aligned} \tag{5.2}$$

The N-particle eigenfunctions of $\hat{\Delta}(\lambda)\equiv\mathrm{Tr}\,\hat{T}(\lambda)$ are taken to be of the form

$$|\psi_N(\{\lambda_j\})\rangle=\prod_{j=1}^{N}B(\lambda_j)|0\rangle \tag{5.3}$$

and the N allowed 'wavenumbers' λ_j satisfy the system of Bethe equations

$$\left(\frac{a(\lambda_l)}{d(\lambda_l)}\right)^{M/2}=\prod_{\substack{j\neq l\\ j=1}}^{N}\frac{f(\lambda_j,\lambda_l)}{f(\lambda_l,\lambda_j)} \tag{5.4}$$

in which f is the element of the R-matrix (3.4) defined in (3.5) and depends on q. The eigenvalues Θ_N of $\hat{\Delta}(\lambda)$ corresponding to these eigenfunctions are

$$\Theta_N(\mu,\{\lambda_j\})=(a(\mu))^{M/2}\prod_{j=1}^{N}f(\mu,\lambda_j)+(d(\mu))^{M/2}\prod_{j=1}^{N}f(\lambda_j,\mu) \tag{5.5}$$

and

$$\hat{\Delta}(\mu)|\psi_N(\{\lambda_j\})\rangle=\Theta_N(\mu,\{\lambda_j\})|\psi_N(\{\lambda_j\})\rangle. \tag{5.6}$$

The eigenfunctions (5.3) are simultaneous eigenfunctions of the momentum operator $\mathcal{P}$ [2]:

$$\mathcal{P}|\psi_N(\{\lambda_j\})\rangle=\exp\left[\mathrm{i}\sum_{j=1}^{N}p(\lambda_j)\right]|\psi_N(\{\lambda_j\})\rangle$$

with

$$p(\lambda)=\mathrm{i}\ln(a(\lambda)/d(\lambda)). \tag{5.7}$$

Since $a(\lambda)$ and $d(\lambda)$ are given explicitly by (5.2) and f is defined by (3.5), the Bethe equations (5.4) finally take the form

$$\left(\frac{[\frac{1}{2}(\Lambda+1)-\mathrm{i}\lambda_l][\frac{1}{2}(\Lambda-1)-\mathrm{i}\lambda_l]}{[\frac{1}{2}(\Lambda+1)+\mathrm{i}\lambda_l][\frac{1}{2}(\Lambda-1)+\mathrm{i}\lambda_l]}\right)^{M/2}=\prod_{\substack{j=1\\ j\neq l}}^{N}\frac{[\mathrm{i}\lambda_l-\mathrm{i}\lambda_j+1]}{[\mathrm{i}\lambda_l-\mathrm{i}\lambda_j-1]} \tag{5.8}$$

while the eigenvalue Θ_N is

$$\Theta_N(\mu,\{\lambda_j\})=(a(\mu))^{M/2}\prod_{j=1}^{N}\frac{[\mathrm{i}\mu-\mathrm{i}\lambda_j-1]}{[\mathrm{i}\mu-\mathrm{i}\lambda_j]}+(d(\mu))^{M/2}\prod_{j=1}^{N}\frac{[\mathrm{i}\lambda_j-\mathrm{i}\mu-1]}{[\mathrm{i}\lambda_j-\mathrm{i}\mu]}. \tag{5.9}$$

The eigenvalue $p(\lambda)$ of the momentum $\mathscr{P}$ is

$$p(\lambda)=\tfrac{1}{2}\mathrm{i}\ln\left\{\frac{[\frac{1}{2}(\Lambda+1)-\mathrm{i}\lambda][\frac{1}{2}(\Lambda-1)-\mathrm{i}\lambda]}{[\frac{1}{2}(\Lambda+1)+\mathrm{i}\lambda][\frac{1}{2}(\Lambda-1)+\mathrm{i}\lambda]}\right\} \tag{5.10}$$

and $-\pi\leqslant p(\lambda)\leqslant\pi$, so $\mathrm{e}^{-\mathrm{i}p(\lambda_l)M}$ is equal to the right-hand side of (5.8).

The eigenfunctions (5.3) of $\hat{\Delta}(\lambda)$ are necessarily eigenfunctions of the Hamiltonian $\hat{H}_0$ finally given in (4.24):

$$\hat{H}_0|\psi_N\rangle=\left(\sum_{j=1}^{N}h(\lambda_j)\right)|\psi_N\rangle \tag{5.11}$$

and

$$h(\lambda)=\frac{q-q^{-1}}{2\ln q}\left\{-\mathrm{i}\frac{\partial}{\partial\mu}\ln\left\{\frac{[\mathrm{i}\lambda-\mathrm{i}\mu-1]}{[\mathrm{i}\lambda-\mathrm{i}\mu]}\right\}\bigg|_{\mu=\nu}+\mathrm{i}\frac{\partial}{\partial\mu}\ln\left\{\frac{[\mathrm{i}\mu-\mathrm{i}\lambda-1]}{[\mathrm{i}\mu-\mathrm{i}\lambda]}\right\}\bigg|_{\mu=\nu^*}\right\}$$
$$+\frac{2}{[\frac{1}{2}(\Lambda+1)][\frac{1}{2}(\Lambda-1)]}. \tag{5.12}$$

This follows from the definition (4.22) of $\hat{H}_0$ together with the explicit form (5.9) of the eigenvalue Θ_N. Then by evaluating the derivatives at $\mu=\nu^*$ and $\mu=\nu$ in (5.12) we find the one-particle dispersion relation $h(\lambda)$ to be

$$h(\lambda)=-1/[\tfrac{1}{2}(\Lambda+1)-\mathrm{i}\lambda][\tfrac{1}{2}(\Lambda-1)-\mathrm{i}\lambda]-1/[\tfrac{1}{2}(\Lambda+1)+\mathrm{i}\lambda][\tfrac{1}{2}(\Lambda-1)+\mathrm{i}\lambda]$$
$$+2/[\tfrac{1}{2}(\Lambda+1)][\tfrac{1}{2}(\Lambda-1)]. \tag{5.13}$$

In order to make contact with the QNLS model [22, 23] in this paper we now choose $q=\mathrm{e}^{\mathrm{i}\gamma}$ $(\gamma\in\mathbb{R})$ only: the case of $q=\mathrm{e}^{\gamma}$ will be analysed elsewhere. Since (5.8) is invariant under $\lambda\to\lambda+\mathrm{i}\pi/\gamma$ (i.e. it is periodic of period $\mathrm{i}\pi\gamma^{-1}$) we identify points λ and $\lambda+\mathrm{i}n\pi/\gamma$ in the complex plane (n is an integer $-\infty<n<+\infty$). Equation (5.8) then possesses solutions with both real values $\lambda\in\mathbb{R}$, and complex values, $\lambda+\mathrm{i}\frac{1}{2}\pi\gamma+\mathrm{i}/2(m-2l+1)+O(\mathrm{e}^{-M})$, $l=1,\ldots,m\geqslant1$, $\lambda\in\mathbb{R}$. The solutions $\lambda\in\mathbb{R}$ and $\lambda+\mathrm{i}\pi/2\gamma$, $\lambda\in\mathbb{R}$ correspond to elementary excitations; the remaining complex solutions correspond to m-particle clusters. For the elementary excitations the first two terms of $h(\lambda)$ are $E_\pm(\lambda)$ where

$$\frac{E_\pm(\lambda)}{\sin^2\gamma}=\pm\frac{4(\cosh 2\gamma\lambda\cos\gamma\Lambda\mp\cos\gamma)}{(\cosh 2\gamma\lambda\cos\gamma\mp\cos\gamma)^2+\sin^2(\gamma\Lambda)\sinh^2(2\gamma\lambda)} \tag{5.14}$$

and $\lambda\in\mathbb{R}$ (upper signs in (5.14) and $\lambda\to\lambda+\mathrm{i}\frac{1}{2}\pi\gamma^{-1}$ with $\lambda\in\mathbb{R}$ (lower signs in (5.14)). We can choose $\Lambda>1$, a choice consistent with the interpretation that Λ is essentially the inverse lattice spacing: this choice ensures that the third term of (5.13) is positive. Finally we choose

$$1<\Lambda<\tfrac{1}{2}\pi\gamma^{-1} \qquad 0<\gamma<\tfrac{1}{2}\pi \tag{5.15}$$

which restricts us to the Bose gas model. Other choices remain for further studies.

For $\lambda \in \mathbb{R}$ and the case of upper signs in (5.14) $E_+(\lambda)$ is minimum at $\lambda=0$ and $E_+(0)=E_+<0$: $E_+(\lambda)$ rises to two symmetric maxima where $E_+(\lambda)>0$ and then falls symmetrically and monotonically to zero as $|\lambda|\to\infty$. In the second case (lower signs in (5.14)), $E_-(0)=E_-<0$ rising symmetrically and monotonically from this minimum to zero as $|\lambda|\to\infty$. Evidently

$$E_+=\frac{4\sin^2\gamma}{\cos\gamma\Lambda-\cos\gamma} \qquad E_-=\frac{-4\sin^2\gamma}{\cos\gamma\Lambda+\cos\gamma} \tag{5.16}$$

and there is the gap

$$\Gamma_0=E_--E_+=\frac{8\cos\gamma\Lambda\sin^2\gamma}{\sin\gamma(\Lambda+1)\sin\gamma(\Lambda-1)}>0. \tag{5.17}$$

The third term in $h(\lambda)$ is

$$2/[\tfrac{1}{2}(\Lambda+1)][\tfrac{1}{2}(\Lambda-1)]=-E_+=-\frac{4\sin^2\gamma}{\cos\gamma\Lambda-\cos\gamma}>0 \tag{5.18}$$

so there is the acoustic branch of $h(\lambda)$ for which $h(0)=0$, $h(\lambda)>0$ $(\lambda\neq 0)$ and $h(\lambda)\to -E_+>0$ as $|\lambda|\to\infty$. Then there is the second, optical, branch for which $h((\mathrm{i}/2)\pi\gamma^{-1})=\Gamma_0>0$, $h(\lambda+(\mathrm{i}/2)\pi\gamma^{-1})>\Gamma_0$ $(\lambda\neq 0)$ and $h(\lambda+(\mathrm{i}/2)\pi\gamma^{-1})\to -E_+>0$ as $|\lambda|\to\infty$. The gap Γ_0 thus defines the energy of a fundamental particle of the theory which we can call the optical q-boson. The existence of the optical branch in the spectrum $h(\lambda)$ is exceptional to the q-boson model: neither the QLNS model nor the continuum quantum NLS model has an optical branch. In the 'box' notation the gap Γ_0 is

$$\Gamma_0=\frac{4(q^{\Lambda}+q^{-\Lambda})}{[\Lambda+1][\Lambda-1]} \qquad (1<\Lambda<\tfrac{1}{2}\pi\gamma^{-1}). \tag{5.19}$$

The remaining solutions $\lambda+\mathrm{i}\tfrac{1}{2}\pi\gamma^{-1}+\mathrm{i}\tfrac{1}{2}(m-2l+1)$, $m\geqslant 2$, $l=1,\ldots,m$, $\lambda\in\mathbb{R}$ correspond to the m-particle clusters. The periodicity of (5.8) with period $\mathrm{i}\pi\gamma^{-1}$ puts a bound on the integer m: $m-1<\pi\gamma^{-1}$. The energies of the m-particle clusters prove to be

$$h_m(\lambda)=-\frac{\sin\gamma m\sin\gamma}{\cosh\gamma(\lambda+\mathrm{i}[(\Lambda+m)/2])\cosh\gamma(\lambda+\mathrm{i}[(\Lambda-m)/2])}$$
$$-\frac{\sin\gamma m\sin\gamma}{\cosh\gamma(\lambda-\mathrm{i}[(\Lambda+m)/2])\cosh\gamma(\lambda-\mathrm{i}[(\Lambda-m)/2])}-mE_+ \tag{5.20}$$

so $h_1(\lambda)$ is the energy of the optical q-boson. Evidently the energies $h_m(\lambda)(m\geqslant 2)$ lie above $h_1(\lambda)$. It can be proved that the m-clusters with $m<\pi/2\gamma$ form a breather-like spectrum. For, for $m<\pi/2\gamma$, the m-clusters are m-particle string-like states forming actual bound states of m q-bosons. All details of these bound states will be given in a following paper.

The ground state of the model which has the Hamiltonian $\hat{H}=\hat{H}_0-\bar{\mu}\hat{N}$ taken in thermodynamic limit $M\to\infty$ is constructed by filling the states with negative energies. We must distinguish two cases—the case of low density and the case of high density.

In the low-density limit the ground state of the model consists of one Fermi sphere. There are two main branches of the excitation spectrum, the acoustic branch of gapless q-bosons and the optical branch. The optical branch consists of q-bosons with the bare gap

$$\Gamma=\Gamma_0-\bar{\mu}=\frac{4(q^{\Lambda}+q^{-\Lambda})}{[\Lambda+1][\Lambda-1]}-\bar{\mu}>0 \tag{5.21}$$

in which $\bar{\mu}$ is the chemical potential and $0\leqslant\bar{\mu}<\Gamma_0$, and there are the bound states with the bare gaps exceeding Γ.

The high-density limit has $\bar{\mu}>\Gamma_0$. It consists of several Fermi spheres, one filled by the acoustic q-bosons, the others filled by the optical q-bosons and by the clusters.

The existence of the several branches of the spectrum is the main difference of the q-Bose gas model from the QLNS and Bose gas models [23, 24]: it leads to some changes in the detail of the asymptotics of the correlation functions [25] although the main features of these are the same since these correlations do not distinguish discrete and continuous models. The latter is obtained in the $q\to 1$ $(\gamma\to 0)$, $\Lambda=4/c\delta$, $\lambda\to\lambda c^{-1}$, $\delta\to 0$ scaling limit as expected since the $q\to 1$ $(\gamma\to 0)$ limit is the QLNS model. However, the result (2.12) shows that the lattice Bose gas must have a continuum limit which is the Bose gas found directly. These two limits are investigated in section 6.

6. Continuum limits of the model

We first take the $q\to 1$ $(\gamma\to 0)$ limit and regain the QLNS model [22, 23]. When $q\to 1$ the commutation relations (2.10) become those of ordinary bosons, $[\beta_n,\beta_m^{\dagger}]=\delta_{nm}$ and $N_n=\beta_n^{\dagger}\beta_n$. For a subsequent continuum limit $\delta\to 0$ we put $\Lambda=4(c\delta)^{-1}$ and $\lambda\to\lambda c^{-1}$, $c>0$, so that the $\hat{L}_n$-operator (4.1) is

$$\hat{L}_n(\lambda)=\frac{2}{\delta c}\begin{bmatrix}(1+(-1)^n\frac{1}{4}c\delta+\frac{1}{2}c\delta N_n-\frac{1}{2}\mathrm{i}\lambda\delta) & -\mathrm{i}\sqrt{c\delta}\,\beta_n^{\dagger}\rho_n\\ \mathrm{i}\sqrt{c\delta}\,\rho_n\beta_n & (1+(-1)^n\frac{1}{4}c\delta+\frac{1}{2}c\delta N_n+\frac{1}{2}\mathrm{i}\lambda\delta)\end{bmatrix} \tag{6.1}$$

with

$$\rho_n=\sqrt{1+\tfrac{1}{4}(-1)^n c\delta+\tfrac{1}{4}c\delta N_n}\,. \tag{6.2}$$

But this is exactly the $\hat{L}_n$-operator of the QLNS model introduced in [22].

The $q\to 1$ limit immediately replaces $f(\mu,\lambda)$ and $g(\mu,\lambda)$ equations (3.5), by

$$f(\mu,\lambda)=1+\frac{\mathrm{i}c}{\mu-\lambda}\qquad g(\mu,\lambda)=\frac{\mathrm{i}c}{\mu-\lambda} \tag{6.3}$$

(we put $\lambda,\mu\to\lambda c^{-1},\mu c^{-1}$ in (3.5)) so the R-matrix (3.4) becomes the rational R-matrix of XXX type. Evidently the Hamiltonian $\hat{H}_0$ becomes that of the QLNS model of [23]

$$\hat{H}_0=-4(3c\delta^3)^{-1}\sum_{n=1}^{M}\{\hat{t}_n+\hat{t}_n^{\dagger}+(8-c\delta)(8-2c\delta)^{-1}\}+4(3\delta^2)^{-1}(1-\tfrac{1}{4}\delta^2c^2)^{-1}\hat{N}. \tag{6.4}$$

The $\hat{t}_n$ are now, for odd sites $n=1$ (mod 2),

$$\hat{t}_n=(g^{\dagger}(n+2)g(n+1))^{-1}\{(g^{\dagger}(n)g(n-1))^{-1}(g^{\dagger}(n+1)g(n))^{-1}$$
$$\times(g^{\dagger}(n+1)\sigma^3g(n-1))\}(g^{\dagger}(n+2)g(n+1)) \tag{6.5}$$

and for even sites $n = 0 \pmod 2$

$$\hat{t}_n = (g^\dagger(n-1)g(n-2))^{-1}\{(g^\dagger(n+1)g(n-1))^{-1}(g^\dagger(n)g(n-1))^{-1} \times (g^\dagger(n+1)\sigma^3 g(n-1))\}(g^\dagger(n-1)g(n-2)) \tag{6.6}$$

and g is the two-component column vector with elements

$$g_1(n) = -\mathrm{i}\beta_n^\dagger \qquad g_2(n) = (N_n + 4c^{-1}\delta^{-1} - 1)^{1/2} \tag{6.7}$$

for $n = 1 \pmod 2$, and

$$g_1(n) = -\mathrm{i}\beta_n^\dagger \qquad g_2(n) = (N_n + 4c^{-1}\delta^{-1})^{1/2} \tag{6.8}$$

for $n = 0 \pmod 2$.

The dispersion relation is

$$h(\lambda) = 4(3\delta^2)^{-1}\{(\delta^2\lambda^2 + c^2\delta^2 - 16)\{(\delta^2\lambda^2 + c^2\delta^2 - 16) + 16\lambda^2c^2\}^{-1} - (c^2\delta^2 - 16)^{-1}\} \tag{6.9}$$

$h(0) = 0$ and there is only one branch and no gap. The momentum is

$$p(\lambda) = \mathrm{i}\ln\left\{\frac{(1 - \frac{1}{4}c\delta - \frac{1}{2}\mathrm{i}\delta\lambda)(1 + \frac{1}{4}c\delta - \frac{1}{2}\mathrm{i}\delta\lambda)}{(1 - \frac{1}{4}c\delta + \frac{1}{4}\mathrm{i}\delta\lambda)(1 + \frac{1}{4}c\delta + \frac{1}{2}\mathrm{i}\delta\lambda)}\right\} \tag{6.10}$$

and the Bethe equations are

$$\left\{\frac{(1 - \frac{1}{4}c\delta - \frac{1}{2}\mathrm{i}\delta\lambda_l)(1 + \frac{1}{4}c\delta - \frac{1}{2}\mathrm{i}\delta\lambda_l)}{(1 - \frac{1}{4}c\delta + \frac{1}{2}\mathrm{i}\delta\lambda_l)(1 + \frac{1}{4}c\delta + \frac{1}{2}\mathrm{i}\delta\lambda_l)}\right\}^{M/2} = \prod_{\substack{j \neq l \\ j=1}}^{N}\left(\frac{\lambda_l - \lambda_j - \mathrm{i}c}{\lambda_l - \lambda_j + \mathrm{i}c}\right). \tag{6.11}$$

All of these results coincide with those of the QLNS model [23]. Equations (6.11) possess only real solutions.

As shown in [23], in the continuum limit $\delta \to 0$ with $x = n\delta$, $L = M\delta$ and $a_n^\dagger \to \sqrt{\delta}\,\beta^\dagger(x)$, $a_n \to \sqrt{\delta}\,\beta(x)$, the Hamiltonian (6.4) becomes exactly that of the repulsive quantum NLS model (the Bose gas) with coupling constant $c > 0$, namely

$$\hat{H}_{\mathrm{BG}} = \int \{\partial_x\beta^\dagger(x)\partial_x\beta(x) + c\beta^\dagger(x)\beta^\dagger(x)\beta(x)\beta(x)\}\,\mathrm{d}x. \tag{6.12}$$

Likewise equations (5.8)–(5.10) take the Bose gas forms [24] $h(\lambda) = \lambda^2$, $p(\lambda) = \lambda$, and

$$\mathrm{e}^{-\mathrm{i}L\lambda_l} = \prod_{\substack{j \neq l \\ j=1}}^{N}\frac{(\lambda_l - \lambda_j - \mathrm{i}c)}{(\lambda_l - \lambda_j + \mathrm{i}c)}. \tag{6.13}$$

The continuum limit just taken on the QLNS model, results (6.4)–(6.11), also takes the commutation relations (2.1) for M q-bosons to (2.12), the usual canonical commutation relations for bosons. It can be shown that in this limit the q-Bose gas Hamiltonian (4.24) turns directly into the Bose gas Hamiltonian (6.12). To investigate this limit it is useful to use the representation of the q-bosons a_n, $a_n^\dagger$ (2.10) in terms of ordinary bosons β_n, $\beta_n^\dagger$, for which $[\beta, \beta_m^\dagger] = \delta_{nm}$. The representation is

$$a_n = \left(\frac{[N_n + 1]}{N_n + 1}\right)^{1/2}\beta_n \qquad a_n^\dagger = \beta_n^\dagger\left(\frac{[N_n + 1]}{N_n + 1}\right)^{1/2} \qquad N_n = \beta_n^\dagger\beta_n. \tag{6.14}$$

In the same limit the R-matrix (3.4) must become the R-matrix of the Bose gas (6.3) and this is the $q \to 1$ ($\gamma \to 0$) limit of (3.4). But by restricting attention to the range

(5.15) for Λ we set $\Lambda = 4(c\delta)^{-1}$ and $\gamma \to \gamma\delta$. By multiplying the $\hat{L}$-operators (4.1) from two adjacent sites, we see that when $\delta \to 0$, the infinitesimal $\hat{L}$-operator is

$$\hat{L}(x) = \hat{L}_{n+1}\hat{L}_n = 1 - \delta\hat{l}(x) + O(\delta^2) \tag{6.15}$$

where $\hat{l}(x)$ is the Lax operator for the continuous quantum nonlinear Schrödinger model (the Bose gas) [28]:

$$\hat{l}(x) = \begin{pmatrix} \lambda/2 & \sqrt{c}\,\beta^\dagger(x) \\ -\sqrt{c}\,\beta(x) & -\lambda/2 \end{pmatrix}.$$

Moreover since $\gamma \to \delta\gamma$ as $\delta \to 0$, the R-matrix (3.4) turns into the R-matrix (6.3).

It follows that all details of the q-Bose gas model reduce to all details of the Bose gas in the natural continuum limit. This includes the dispersion relation which becomes $h(\lambda) = \lambda^2$, and which has a single acoustic branch and is gapless.

7. Summary and conclusions

We have constructed the Hamiltonian $\hat{H}_0$ (4.24) of the quantum integrable q-Bose gas model and solved it for its eigenstates (equation (5.3)) and eigenvalues (equation (5.11)) which are defined through the dispersion relations $h(\lambda)$, equation (5.13). We have used the method of 'projectors' [22] to calculate $\hat{H}_0$, but there exists another approach, based on the notion of the fundamental R-matrix [29]. We shall return to this alternative approach in another paper. The advantage of the 'projector' method is that the Hamiltonian (4.22) which is obtained is formulated in terms of initial local q-Bose fields.

The main physical result of the paper is that the q-deformed boson model has both an acoustic branch and optical branches in its spectrum. The second result is that, as projected from the 'continuum limit' of a set of M q-bosons, the $\delta \to 0$ continuum limit of the q-boson model is the repulsive nonlinear Schrödinger model (the Bose gas). We believe this exemplifies the position surrounding continuum field theories—namely that quantization of these by canonical or by quantum group quantization leads to the same quantum field theory. This is not necessarily true for the discrete models with a finite number or a countably infinite number of degrees of freedom, and the q-deformed boson (q-boson model) solved in this paper exemplifies the latter. Calculations for the correlation functions of the q-Bose gas model treated in this paper will be reported in a following paper [25]. Finally we note that the R-matrix (3.4) is the R-matrix of the sine-Gordon model as given in [22]. Thus we can expect to derive the related quantum sine-Gordon lattice model. This investigation will be reported elsewhere, while details of the string-like bound states (section 5) will also be given in a following paper.

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